

# The class three groups of order $p^9$ with exponent $p$

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February 2017

## Abstract

In this note we give a complete list of all class two groups of prime exponent with order  $p^k$  for  $k \leq 8$ . For every group in this list we are able to show that the number of conjugacy classes of the group is polynomial in  $p$ , and that the order of the automorphism group is also polynomial in  $p$ . In addition, for each group in the list, the number of class 3 descendants of order  $p^9$  with exponent  $p$  is PORC. So the total number of class 3 groups of order  $p^9$  with exponent  $p$  is PORC.

## 1 Introduction

In 1960 Graham Higman wrote two immensely important and significant papers [3], [4]. In these papers he conjectured that if we set  $f(p, n)$  equal to the number of groups of order  $p^n$ , then  $f(p, n)$  is a PORC function of  $p$  for each positive integer  $n$  — this is known as Higman's PORC conjecture. (A function  $f(p)$  is said to be PORC, **P**olynomial **O**n **R**esidue **C**lasses, if there is a finite set of polynomials in  $p$ ,  $g_1(p), g_2(p), \dots, g_k(p)$ , and a fixed integer  $M$ , such that for each prime  $p$ ,  $f(p) = g_i(p)$  for some  $i$  ( $1 \leq i \leq k$ ), with the choice of  $i$  depending on the residue class of  $p \bmod M$ .) He also proved that the number of  $p$ -class two groups of order  $p^n$  is a PORC function of  $p$  for each  $n$ . At the time Higman made his conjecture it was known to hold true for  $n \leq 5$ , and it has since been proved correct for  $n \leq 7$  (see [7], [9]). For example, for  $p \geq 5$  the number of groups of order  $p^6$  is

$$3p^2 + 39p + 344 + 24 \gcd(p-1, 3) + 11 \gcd(p-1, 4) + 2 \gcd(p-1, 5).$$

The number of groups of order  $p^8$  with exponent  $p$  has also been shown to be PORC (see [12]). For reasons that I will explain below I became interested

in the question of whether or not the number of class 3 groups of order  $p^9$  with exponent  $p$  is PORC. These groups are all immediate descendants of class two groups of exponent  $p$  with order dividing  $p^8$ . (A  $p$ -group  $H$  is said to be an immediate descendant of a  $p$ -group  $G$  if  $G \cong H/N$ , where  $N$  is the last non-trivial term of the lower  $p$ -central series of  $H$ . See [7] for details.) In [11] I gave a complete list of the 70 class two groups of exponent  $p$  ( $p > 2$ ), together with a description of their automorphism groups. In this paper I consider each of these 70 groups in turn, and for each group give the number of class 3 descendants of order  $p^9$  with exponent  $p$ . (The numbers given are valid for all  $p \geq 5$ .) In each case the number of descendants is PORC, and so:

**Theorem 1** *The number of class three groups of order  $p^9$  with exponent  $p$  is PORC.*

However there are finite  $p$ -groups for which the number of immediate descendants of given order is *not* PORC. For example du Sautoy and Vaughan-Lee [2] give an example of a class two group  $G$  with exponent  $p$  and order  $p^9$  with the following properties:

1. The number of conjugacy classes of  $G$  is not PORC,
2. The order of the automorphism group of  $G$  is not PORC,
3. The number of immediate descendants of  $G$  with exponent  $p$  and order  $p^{10}$  is not PORC.

This example shows that at least one group of order  $p^9$  has a non-PORC number of descendants of order  $p^{10}$ . However this does not disprove Higman's PORC conjecture for groups of order  $p^{10}$ . If we were to follow the methods of [7] and [9] in an attempt to settle the conjecture for  $p^{10}$  we would need to draw up a complete and irredundant list of the groups with order dividing  $p^9$ , and compute the number of descendants of order  $p^{10}$  of each group in the list. There are likely to be other groups of order  $p^9$  with a non-PORC number of descendants of order  $p^{10}$ , so the grand total of groups of order  $p^{10}$  could be PORC, with individual non-PORC summands cancelling out. Nevertheless du Sautoy and I confidently asserted: "*It seems likely that there are other groups of order  $p^9$  with a non-PORC number of immediate descendants of order  $p^{10}$ , and so it is possible that the grand total is PORC, even though not all of the summands are PORC. The authors' own view is that this is extremely unlikely.*"

It is still an open question whether the number of groups of order  $p^{10}$  is PORC, but nevertheless this assertion is now looking rather foolish! Seungjai Lee [5] has recently found a class two group  $H$  with exponent  $p$  and order  $p^8$  with the following properties:

1. The number of conjugacy classes of  $H$  is not PORC,
2. The order of the automorphism group of  $H$  is not PORC,
3. The number of immediate descendants of  $H$  with exponent  $p$  and order  $p^9$  is not PORC.

However, in this case we now know that the total number of class three groups of order  $p^9$  with exponent  $p$  is PORC. In fact, in apparent contradiction to Lee's result, for every single group  $G$  in my list of 70 class two groups of exponent  $p$  with order dividing  $p^8$ :

1. The number of conjugacy classes of  $G$  is a polynomial in  $p$ ,
2. The order of the automorphism group of  $G$  is a polynomial in  $p$ ,
3. The number of class 3 descendants of  $G$  with order  $p^9$  and exponent  $p$  is PORC.

The explanation for this “contradiction” is that Lee's group  $H$  is really a family of groups, one for each  $p$ , as are the groups in my list. For any given  $p$ , Lee's group  $H$  must lie in one of my families, but it does not have to lie in the same family for every  $p$ . It turns out that there are four (families of) groups in my list  $A$ ,  $B$ ,  $C$ ,  $D$ . If  $p = 3$  then  $H \cong A$ , if  $p \equiv 2 \pmod{3}$  then  $H \cong B$ , if  $p \equiv 1 \pmod{3}$  and  $t^3 - 2$  has no roots in  $\text{GF}(p)$  then  $H \cong C$ , and if  $p \equiv 1 \pmod{3}$  and  $t^3 - 2$  has three roots in  $\text{GF}(p)$  then  $H \cong D$ . The non-PORC properties of Lee's group arise from the fact that the number of roots of  $t^3 - 2$  over  $\text{GF}(p)$  is not PORC. (There is a deep theorem in Number Theory that asserts that if we have a polynomial with integer coefficients then the number of roots of the polynomial over  $\text{GF}(p)$  is a PORC function of  $p$  if and only if the Galois group of the polynomial over the rationals is abelian. The Galois group of  $t^3 - 2$  over the rationals is the symmetric group of degree 3.)

## 2 Methods

The groups listed here correspond to Lie algebras over  $\text{GF}(p)$  under the Lazard correspondence. All the presentations are for class two groups of exponent  $p$  and the presentations can also be read as presentations for class two Lie algebras over  $\text{GF}(p)$ , except that products have to read as sums. For example group 8.4.4 below has presentation

$$\langle a, b, c, d \mid [d, b][c, a], [d, c][b, a]^\omega \rangle.$$

This corresponds to the Lie algebra with presentation

$$\langle a, b, c, d \mid [d, b] + [c, a], [d, c] + \omega[b, a] \rangle$$

using  $[\cdot, \cdot]$  to denote the Lie product, or (as I prefer)

$$\langle a, b, c, d \mid db + ca, dc + \omega ba \rangle$$

using juxtaposition to denote the Lie product. (Here, and throughout this paper,  $\omega$  denotes a fixed integer which is primitive modulo  $p$ .) All my calculations are carried out in Lie algebras, and much of the detailed information in this paper is given in Lie algebra notation. I make no apology for mixing up group and Lie algebra notation in this way. Theorem 1 is both a theorem in groups and in Lie algebras. A group theorist who wants to look at the groups in this paper would probably prefer to see presentations in group notation, and so this is the way I have given them. But the calculations are all in Lie algebras, and it would be awkward to translate their details back into group notation.

I have computed the number of descendants of each group using the Lie algebra version of the  $p$ -group generation algorithm (see Newman [6] and O'Brien [8]). Let  $L$  be a class two Lie algebra over  $\text{GF}(p)$ . The first step is to compute the  $p$ -covering algebra which is the largest class 3 Lie algebra  $K$  with an ideal  $M$  such that

1.  $K/M \cong L$ ,
2.  $M \leq \zeta(K) \cap K^2$ .

(Here  $\zeta(K)$  denotes the centre of  $K$ .) It is quite easy to construct  $K$  with the prime  $p$  symbolic. (So in effect we construct a family of covering algebras, one for each  $p$ .) The ideal  $M$  is called the  $p$ -multiplier of  $L$ , and  $N = K^3 \leq M$  is called the nucleus of  $L$ . An allowable subalgebra of  $M$  is

a proper subalgebra  $S$  such that  $S + N = M$ . The immediate descendants of  $L$  are the Lie algebras  $L/S$  where  $S$  is an allowable subalgebra. The automorphism group of  $L$  acts on  $M$ , and two descendants  $L/S$ ,  $L/T$  are isomorphic if and only if  $S$  and  $T$  are in the same orbit under the action of the automorphism group. Since we are only calculating descendants of dimension 9, if  $L$  has dimension  $n$  then we can restrict our attention to allowable subalgebras of  $M$  of codimension  $9 - n$ . To compute the number of immediate descendants of  $L$  of dimension 9 we need to compute the number of orbits of allowable subspaces of  $M$  of codimension  $9 - n$  under the action of the automorphism group. We mostly do this in two steps. If  $S$  is an allowable subalgebra of codimension  $9 - n$  in  $M$ , then  $S \cap N$  has codimension  $9 - n$  in  $N$ . So first we compute a set of representatives for the orbits of subspaces of codimension  $9 - n$  in  $N$ . We can restrict our attention to allowable subalgebras of  $M$  such that  $S \cap N$  is one of these representatives. If  $T$  is one of these representatives, and if  $S \cap N = T$ , then

$$K/S \cong (K/T)/(S/T).$$

Furthermore the orbits of allowable subalgebras  $S$  with  $S \cap N = T$  correspond to orbits under the stabilizer of  $T$  of subalgebras of codimension  $9 - n$  in  $M/T$  which have trivial intersection with  $N/T$ . For most of the Lie algebras with descendants of dimension 9 we give the dimensions of  $M$  and  $N$ , and a set of representatives for the orbits of subspaces of  $N$  of codimension  $9 - n$ . Then for each such representative  $T$  we give the number of descendants  $K/S$  where  $S \cap N = T$ .

In a few cases I use Higman's ideas to compute the number of orbits of subalgebras of  $M$  of codimension  $9 - n$ , and then obtain the number of descendants by counting the number of orbits of non-allowable subalgebras.

### 3 Order $p^3$

#### Group 3.2.1

$$\langle a, b \rangle$$

(There are no relations here, since this is the free class two group of exponent  $p$  on two generators. As mentioned above, all the presentations are for class two groups of exponent  $p$ .) The number of conjugacy classes is  $p^2 + p - 1$ , and the automorphism group has order  $(p^2 - 1)(p^2 - p)p^2$ . There are no class 3 descendants of order  $p^9$ .

## 4 Order $p^4$

### Group 4.3.1

$$\langle a, b, c \mid [c, a], [c, b] \rangle$$

The number of conjugacy classes is  $p^3 + p^2 - p$ , and the automorphism group has order  $(p-1)(p^2-1)(p^2-p)p^5$ . There are no class 3 descendants of order  $p^9$ .

## 5 Order $p^5$

### 5.3 Three generator groups

#### Group 5.3.1

$$\langle a, b, c \mid [c, b] \rangle$$

The number of conjugacy classes is  $2p^3 - p$ , and the automorphism group has order  $(p-1)(p^2-1)(p^2-p)p^8$ . This group has 12 class 3 descendants of order  $p^9$  and exponent  $p$ .

### 5.4 Four generator groups

#### Group 5.4.1

$$\langle a, b, c, d \mid [c, a], [c, b], [d, a], [d, b], [d, c] \rangle$$

The number of conjugacy classes is  $p^4 + p^3 - p^2$ , and the automorphism group has order  $(p^2-1)^2(p^2-p)p^8$ . There are no class 3 descendants of order  $p^9$ .

#### Group 5.4.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle$$

The number of conjugacy classes is  $p^4 + p - 1$ , and the automorphism group has order  $(p^5-p)(p^5-p^4)(p^3-p)p^2$ . There are no class 3 descendants of order  $p^9$ .

## 6 Order $p^6$

### 6.3 Three generator groups

#### Group 6.3.1

$$\langle a, b, c \rangle$$

The number of conjugacy classes is  $p^4 + p^3 - p$ , and the automorphism group has order  $(p^3 - 1)(p^3 - p)(p^3 - p^2)p^9$ . This group has

$$\begin{aligned} & p^7 + p^6 + 3p^5 + 5p^4 + 11p^3 + (22 + 2 \gcd(p - 1, 3))p^2 \\ & + (46 + 5 \gcd(p - 1, 3) + \gcd(p - 1, 4))p \\ & + 60 + 6 \gcd(p - 1, 3) + 2 \gcd(p - 1, 4) \end{aligned}$$

class 3 descendants of order  $p^9$  and exponent  $p$ . In this group the  $p$ -multiplier  $M$  is equal to the nucleus  $N$ , and so the number of descendants is equal to the number of orbits of subspaces of  $N$  of codimension 3.

### 6.4 Four generator groups

#### Group 6.4.1

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b], [d, c] \rangle$$

The number of conjugacy classes is  $2p^4 - p^2$  and the order of the automorphism group is  $(p - 1)^2(p^2 - 1)(p^2 - p)p^{13}$ . The nucleus has dimension 5.

We can assume that one of the following 16 sets of relations holds:

$$\begin{aligned}
& bab = bac = 0, \\
& bac = cac + bab = 0, \\
& bac = cac + \omega bab = 0, \\
& caa = bab = 0, \\
& caa = bab - cac = 0, \\
& caa = bab - \omega cac = 0, \\
& caa = bac = 0, \\
& caa = cac = 0, \\
& bac = caa - bab = 0, \\
& cac = caa - bab = 0, \\
& baa = caa = 0, \\
& baa = caa - bab = 0, \\
& baa = caa - cac = 0, \\
& baa = caa - bab + cac = 0, \\
& baa = caa - bab + \omega cac = 0, \\
& baa - cac = caa - \alpha bab - cac = 0,
\end{aligned}$$

for some  $\alpha$  with  $x^3 + \alpha^{-1}x - \alpha^{-1}$  irreducible over  $\text{GF}(p)$ . (Throughout this paper  $\omega$  denotes a fixed integer which is primitive modulo  $p$ .) The number of immediate descendants of order  $p^9$  with exponent  $p$  in the 16 cases are as



follows:

$$\begin{aligned}
& 7p + 50 + \gcd(p - 1, 4), \\
& \frac{1}{2}(3p^2 + 17p + 80 + \gcd(p - 1, 4)), \\
& \frac{1}{2}(3p^2 + 11p + 30 + \gcd(p - 1, 4)), \\
& p^4 + 3p^3 + 7p^2 + 20p + 69 + 2 \gcd(p - 1, 3) + \gcd(p - 1, 4), \\
& \frac{1}{2}(p^5 + 2p^4 + 4p^3 + 11p^2 + 23p + 31) + \gcd(p - 1, 4), \\
& \frac{1}{2}(p^5 + 2p^4 + 4p^3 + 11p^2 + 23p + 31) + \gcd(p - 1, 4), \\
& p^3 + 3p^2 + 12p + 55 + 6 \gcd(p - 1, 3) + 3 \gcd(p - 1, 4) + \gcd(p - 1, 5), \\
& 2p^2 + 9p + 35 + \gcd(p - 1, 3), \\
& p^4 + 2p^3 + 5p^2 + 10p + 17 + (p + 5) \gcd(p - 1, 3) \\
& \quad + (p + 3) \gcd(p - 1, 4) + \gcd(p - 1, 5), \\
& p^3 + 2p^2 + 8p + 17 + (p + 3) \gcd(p - 1, 3) + \gcd(p - 1, 4), \\
& p^2 + 3p + 23 + \gcd(p - 1, 3), \\
& p^4 + 2p^3 + 4p^2 + 8p + 9 + (p + 2) \gcd(p - 1, 3) + \gcd(p - 1, 4), \\
& p^5 + 2p^4 + 3p^3 + 5p^2 + 10p + 9 + (p + 1) \gcd(p - 1, 3), \\
& \frac{1}{6}(p^6 + p^5 + p^4 + 4p^3 + 9p^2 + 7p + 13 + 2(p + 1) \gcd(p - 1, 3)), \\
& \frac{1}{2}(p^6 + p^5 + p^4 + 2p^3 + 3p^2 + 3p + 5), \\
& \frac{1}{3}(p^6 + p^5 + p^4 + p^3 + 3p^2 + p + 4 + 2(p + 1) \gcd(p - 1, 3)).
\end{aligned}$$

### Group 6.4.2

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [b, a], [d, c] \rangle$$

The number of conjugacy classes is  $p^4 + 2p^3 - p^2 - 2p + 1$  and the order of the automorphism group is  $2(p^2 - 1)^2(p^2 - p)^2p^8$ . The nucleus has dimension 4, and is spanned by  $baa$ ,  $bab$ ,  $caa$ ,  $cac$ . There are two orbits of the automorphism group on the one dimensional subspaces of the nucleus, and we take the (single) relation on the nucleus to be  $cac = 0$  or  $cac = bab$ . The number

of class 3 descendants of order  $p^9$  with exponent  $p$  in the two cases is

$$\begin{aligned} & (p+11) \gcd(p-1, 3) + 26, \\ & \frac{1}{2}(p^3 + 3p^2 + 11p + 26) \gcd(p-1, 3) + \frac{1}{2}(p^2 + 3p + 29) \\ & + \frac{1}{2} \gcd(p-1, 4) + \gcd(p-1, 9) + \gcd(p-1, 12). \end{aligned}$$

### Group 6.4.3

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] \rangle$$

The number of conjugacy classes is  $p^4 + p^3 - p$  and the order of the automorphism group is  $(p-1)(p^2-1)(p^2-p)p^{12}$ . The nucleus has dimension 4, and is spanned by  $baa$ ,  $bab$ ,  $bac$  and  $bad$ . There are two orbits of the automorphism group on the one dimensional subspaces of the nucleus, and we take the (single) relation on the nucleus to be  $bab = 0$  or  $bad = 0$ . The number of class 3 descendants of order  $p^9$  with exponent  $p$  in the the first case is

$$p^2 + 6p + 32 + 5 \gcd(p-1, 3) + 2 \gcd(p-1, 4),$$

and in the second case it is

$$\begin{aligned} & 75 \text{ if } p = 5, \\ & p^2 + 5p + 22 + \gcd(p-1, 3) + \gcd(p-1, 5) \text{ if } p > 5. \end{aligned}$$

### Group 6.4.4

$$\langle a, b, c, d \mid [c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega \rangle$$

The number of conjugacy classes is  $p^4 + p^2 - 1$  and the order of the automorphism group is  $2(p^4-1)(p^4-p^2)p^8$ . The nucleus has dimension 4, and is spanned by  $baa$ ,  $bab$ ,  $bac$  and  $bad$ . But there is only one orbit of the automorphism group on the the one dimensional subspaces of the nucleus, and so we can take the single relation on the nucleus to be  $bad = 0$ . The number of class 3 descendants of order  $p^9$  with exponent  $p$  is

$$\frac{1}{2}(4p^3 + 5p^2 + 23p + 35) - \frac{1}{2}(p^3 + p^2 + 5p + 4) \gcd(p-1, 3) + \frac{1}{2} \gcd(p-1, 4).$$

## 6.5 Five generator groups

### Group 6.5.1

$$\langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p^3$  and the order of the automorphism group is  $(p^2 - 1)(p^2 - p)(p^3 - 1)(p^3 - p)(p^3 - p^2)p^{11}$ . There are no class 3 descendants of order  $p^9$ .

### Group 6.5.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle \times \langle e \rangle$$

The number of conjugacy classes is  $p^5 + p^2 - p$  and the order of the automorphism group is  $(p^6 - p^2)(p^6 - p^5)(p^4 - p^2)p^3(p^2 - p)$ . There are no class 3 descendants of order  $p^9$ .

## 7 Order $p^7$

### 7.4 Four generator groups

#### Group 7.4.1

$$\langle a, b, c, d \mid [c, b], [d, b], [d, c] \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p^2$ , and the automorphism group has order  $(p - 1)(p^3 - 1)(p^3 - p)(p^3 - p^2)p^{15}$ . The nucleus,  $N$ , has dimension 9, and is spanned by

$$bab, bac, bad, cac, cad, dad, baa, caa, daa.$$

The subspace

$$B = \langle bab, bac, bad, cac, cad, dad \rangle$$

of the nucleus is invariant under the automorphism group. The nucleus has

$$p^2 + 4p + 76 + 3 \gcd(p - 1, 3) + 2 \gcd(p - 1, 4)$$

orbits of subspaces of dimension 7, and we organize these according to their intersection with  $B$ . This intersection will have dimension 4, 5 or 6.

There is one orbit of subspaces of  $N$  of dimension 7 containing  $B$ , and an orbit representative is

$$\langle bab, bac, bad, cac, cad, dad, baa \rangle.$$

There are  $p + 8$  descendants of group 7.4.1 of order  $p^9$  with exponent  $p$  satisfying relations

$$bab = bac = bad = cac = cad = dad = baa = 0.$$

There are 4 orbits of subspaces of dimension 5 in  $B$ . Orbit representatives are

$$\begin{aligned} &\langle bab, bac, bad, cac, cad \rangle, \\ &\langle bab, bac, bad, cac, dad \rangle, \\ &\langle bab, bac, bad, \omega cac + dad, cad \rangle, \\ &\langle bab, bac, cac + bad, cad, dad \rangle. \end{aligned}$$

The corresponding orbits are (respectively) the intersections with  $B$  of 4, 5, 3 and 3 orbits of subspaces of dimension 7 in  $N$ . The 4 orbit representatives corresponding to the first of these four are

$$\begin{aligned} &\langle bab, bac, bad, cac, cad, baa, caa \rangle, \\ &\langle bab, bac, bad, cac, cad, baa, caa - dad \rangle, \\ &\langle bab, bac, bad, cac, cad, baa, daa \rangle, \\ &\langle bab, bac, bad, cac, cad, baa - dad, daa \rangle, \end{aligned}$$

and these yield (respectively) 5,  $p + 5$ , 10, 8 descendants of order  $p^9$  with exponent  $p$ . For the other 3 orbit representatives of subspaces of dimension 5 in  $B$  we obtain the following orbit representatives for subspaces dimension 7 in  $B$ :

$$\begin{aligned} &\langle bab, bac, bad, cac, dad, baa, caa \rangle, \\ &\langle bab, bac, bad, cac, dad, baa - cad, caa \rangle, \\ &\langle bab, bac, bad, cac, dad, baa, caa + daa \rangle, \\ &\langle bab, bac, bad, cac, dad, baa - cad, caa + daa \rangle, \\ &\langle bab, bac, bad, cac, dad, caa, daa \rangle, \end{aligned}$$

yielding 5, 5,  $(p + 7)/2$ ,  $(p + 7)/2$ , 6 descendants respectively;

$$\begin{aligned} &\langle bab, bac, bad, \omega cac + dad, cad, baa, caa \rangle, \\ &\langle bab, bac, bad, \omega cac + dad, cad, baa - cac, baa \rangle, \\ &\langle bab, bac, bad, \omega cac + dad, cad, caa, daa \rangle, \end{aligned}$$

yielding  $(p+7)/2$ ,  $(p+7)/2$ , 4 descendants respectively, and:

$$\begin{aligned} &\langle bab, bac, cac + bad, cad, dad, baa, caa \rangle, \\ &\langle bab, bac, cac + bad, cad, dad, baa, daa \rangle, \\ &\langle bab, bac, cac + bad, cad, dad, caa, 2\omega baa + daa \rangle, \end{aligned}$$

yielding 6,  $p+5$ ,  $p+3$  descendants respectively.

The 15 orbits of subspaces of  $B$  of dimension 4 have representatives

$$\begin{aligned} &\langle bab, bac, bad, cac \rangle, \\ &\langle bab, bac, bad, cad \rangle, \\ &\langle bab, bac, bad, \alpha cac + cad - \alpha dad \rangle \text{ (} 1 + 4\alpha^2 \text{ not square)}, \\ &\langle bab, bac, cac, dad \rangle, \\ &\langle bab, bac, cad, dad \rangle, \\ &\langle bab, bac, cac - dad, cad \rangle, \\ &\langle bab, bac, \omega cac - dad, cad \rangle, \\ &\langle bab, bac, bad - cac, cad \rangle, \\ &\langle bab, bac, bad - cac, dad \rangle, \\ &\langle bab, cac, cad - bad, dad - bad \rangle, \\ &\langle bab, bad - cac, cad, \omega bad + dad \rangle, \\ &\langle bab, bad - cac, cad, \beta bac + bad - dad \rangle, \\ &\langle bac, bad, cad, \omega cac + dad \rangle, \\ &\langle bac, bad, cad, bab + \omega cac - dad \rangle, \\ &\langle bac, bad, bab - cac + \gamma cad, bab - \gamma cad - dad \rangle, \end{aligned}$$

where  $\beta$  is chosen so that  $x^3 + x - \beta$  is irreducible over  $\text{GF}(p)$ , and where  $\gamma$  is chosen so that  $1 + \gamma^2$  is not square modulo  $p$ .

The first of these 15 subspaces gives 6 orbits of subspaces of dimension 7 in  $N$ , with representatives of the form

$$\langle bab, bac, bad, cac, baa - xcad - ydad, caa - zcad - tdad \rangle.$$

These 6 representatives together yield 20 descendants of order  $p^9$  with exponent  $p$ .

The second of these 15 subspaces gives  $7 + (\gcd(p-1, 3) - 1)/2$  orbits of subspaces of dimension 7 in  $N$ , with representatives of the form

$$\langle bab, bac, bad, cad, baa - xcac - ydad, caa - zdad, daa - tcac \rangle.$$

Between them these representatives yield  $\frac{1}{2}(p + 35 + (p + 9) \gcd(p - 1, 3))$  descendants of order  $p^9$  with exponent  $p$ .

The third gives  $5 - (\gcd(p - 1, 3) - 1)/2$  orbits of subspaces of dimension 7 in  $N$ , with representatives of the form

$$\langle bab, bac, bad, \omega cac + cad - \omega dad, baa - xcac - ydad, caa - zcac - tdad, daa \rangle.$$

Between them these representatives yield  $\frac{1}{2}(5p + 25 - (p + 3) \gcd(p - 1, 3))$  descendants of order  $p^9$  with exponent  $p$ .

I haven't forgotten the fourth!!!

The fifth gives  $5 + \gcd(p - 1, 3)$  orbits of subspaces of dimension 7 in  $N$ , with representatives of the form

$$\langle bab, bac, cad, dad, baa - xcac, caa - ybad, daa - zcac \rangle.$$

Each of these representatives yields exactly 2 descendants of order  $p^9$  with exponent  $p$ .

The eighth gives 4 orbits of subspaces of dimension 7 in  $N$ , with representatives of the form

$$\langle bab, bac, bad - cac, cad, baa - xdad, caa - ydad, daa - zdad \rangle.$$

Each of these representatives yields exactly 2 descendants of order  $p^9$  with exponent  $p$ .

The thirteenth gives  $3 + \gcd(p - 1, 3)$  orbits of subspaces of dimension 7 in  $N$ , with representatives of the form

$$\langle bac, bad, cad, \omega cac + dad, baa - xcac, caa - ybab, daa - zbab \rangle.$$

Each of these representatives yields exactly 2 descendants of order  $p^9$  with exponent  $p$ .

So far we have accounted for  $46 + 2 \gcd(p - 1, 3)$  of the

$$p^2 + 4p + 76 + 3 \gcd(p - 1, 3) + 2 \gcd(p - 1, 4)$$

orbits of subspaces of dimension 7 in the nucleus  $N$ . Each of the remaining orbits contribute one descendant of order  $p^9$  with exponent  $p$ . So the total number of descendants is

$$p^2 + 13p + 188 + 8 \gcd(p - 1, 3) + 2 \gcd(p - 1, 4).$$

## Group 7.4.2

$$\langle a, b, c, d \mid [d, a], [d, b], [d, c] \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p^2$ , and the automorphism group has order  $(p-1)(p^3-1)(p^3-p)(p^3-p^2)p^{15}$ . The nucleus  $N$  has dimension 8, and is spanned by  $baa, bab, bac, caa, cab, cac, cbb, cbc$ . The number of orbits of subspaces of  $N$  of dimension 6 under the automorphism group is

$$p^4 + p^3 + 5p^2 + 13p + (2p+3) \gcd(p-1, 3) + \gcd(p-1, 4) + 26.$$

The multiplier  $M$  has dimension 11, and the number of orbits of the automorphism group on subspaces of  $M$  of dimension 9 is

$$\begin{aligned} & p^6 + 2p^5 + 6p^4 + 11p^3 + 27p^2 + 77p + 162 + \left(p^2 + \frac{11}{2}p + \frac{27}{2}\right) \gcd(p-1, 3) \\ & + \left(p^2 + \frac{1}{2}p + \frac{3}{2}\right) \gcd(p+1, 3) + (p+6) \gcd(p-1, 4) + \gcd(p-1, 5). \end{aligned}$$

However not all these subspaces are allowable. (The allowable subspaces intersect  $N$  in a subspace of dimension 6.) If a subspace of  $M$  of dimension 9 is not allowable, then it either contains  $N$ , or intersects  $N$  in a subspace of dimension 7.

There is exactly one orbit of subspaces of  $M$  of dimension 9 which contains  $N$ .

There are  $p+4 + \gcd(p-1, 3)$  orbits of subspaces of  $N$  of dimension 7. Representatives for these orbits are as follows:

$$\begin{aligned} & \langle bab, bac, caa, cab, cac, cbb, cbc \rangle, \\ & \langle baa, bac, caa, cab, cac + bab, cbb, cbc \rangle, \\ & \langle baa, bac, caa, cab, cac + \omega bab, cbb, cbc \rangle, \\ & \langle baa, bac, caa - bab, cab, cac, cbb, cbc \rangle, \\ & \langle bab, bac, caa, cab, cac - baa, cbb - baa, cbc - \lambda baa \rangle, \\ & \langle bab, bac, caa, cab, cac - baa, cbb - \omega baa, cbc - \lambda baa \rangle \ (p \equiv 1 \pmod{3}), \\ & \langle bab, bac, caa, cab, cac - baa, cbb - \omega^2 baa, cbc - \lambda baa \rangle \ (p \equiv 1 \pmod{3}), \\ & \langle bab - baa, bac, caa - baa, cab, cac + baa, cbb + baa, cbc - baa \rangle, \end{aligned}$$

where in the subspaces 5,6 and 7 we have  $\lambda = 0$ , or  $\lambda$  running over a set of representatives for equivalence classes of non-zero elements in  $\text{GF}(p)$  under the equivalence relation  $\lambda \sim \mu$  if  $\lambda^3 = \mu^3$ .

There are 7 orbits of subspaces of  $M$  of dimension 9 which intersect  $N$  in the first of these subspaces. There are 7 orbits of subspaces of  $M$  of dimension 9 which intersect  $N$  in the second of these subspaces, and 4 orbits which intersect  $N$  in the third, and there are 5 orbits of subspaces of  $M$  of dimension 9 which intersect  $N$  in the fourth subspace.

When counting orbits of subspaces of  $M$  of dimension 9 which intersect in the fifth, sixth and seventh of these subspaces we need to distinguish between the cases  $p = 1 \bmod 3$  and  $p = 2 \bmod 3$ . When  $p = 2 \bmod 3$  we have  $p$  different subspaces parametrized by  $\lambda$  with  $\lambda \in \text{GF}(p)$ . The number of orbits then depends on the number of roots in  $\text{GF}(p)$  of the polynomial  $x^3 - \lambda x + 1$ . If there are no roots then the number is 1, if there is one root the number is 3, if there are two roots the number is 5, and if there are three roots the number is 7. For most integers  $\lambda$  the number of roots in  $\text{GF}(p)$  of  $x^3 - \lambda x + 1$  is *not* PORC! But as  $\lambda$  ranges over  $\text{GF}(p)$  the numbers of occurrences of 0, 1, 2 or 3 roots is predictable — they occur  $\frac{p+1}{3}$ ,  $\frac{p-1}{2}$ , 1,  $\frac{p-5}{6}$  times respectively. When  $p = 1 \bmod 3$  and  $\lambda = 0$  there are 3 orbits of subspaces of  $M$  of dimension 9 which intersect  $N$  in the fifth subspace, 1 orbit which intersect  $N$  in the sixth subspace and 1 orbit which intersect  $N$  in the seventh subspace. When  $p = 1 \bmod 3$  and  $\lambda \neq 0$  we have  $p-1$  distinct subspaces of the form 5, 6 or 7. Again the number of orbits of subspaces of  $M$  of dimension 9 which intersect  $N$  in one of these subspaces depends on the number of roots of the polynomials  $x^3 - \lambda x + 1$ ,  $x^3 - \lambda x + \omega$ ,  $x^3 - \lambda x + \omega^2$ . As above the numbers of orbits are 1, 3, 5 or 7, with the total number of occurrences of these numbers being  $\frac{p-1}{3}$ ,  $\frac{p-1}{2}$ , 1,  $\frac{p-7}{6}$ .

It is perhaps worth exploring exactly what is happening here. Corresponding to subspaces 5, 6 and 7 we have a family of algebras of dimension 7:

$$\begin{aligned} &\langle a, b, c \mid bab, bac, caa, cab, cac - baa, cbb - baa, cbc - \lambda baa, \text{ class } 3 \rangle, \\ &\langle a, b, c \mid bab, bac, caa, cab, cac - baa, cbb - \omega baa, cbc - \lambda baa, \text{ class } 3 \rangle, \\ &\langle a, b, c \mid bab, bac, caa, cab, cac - baa, cbb - \omega^2 baa, cbc - \lambda baa, \text{ class } 3 \rangle. \end{aligned}$$

(The last two of these presentations only arise when  $p = 1 \bmod 3$ .) The automorphism groups of these algebras depend on the numbers of roots of the polynomials  $x^3 - \lambda x + 1$ ,  $x^3 - \lambda x + \omega$ ,  $x^3 - \lambda x + \omega^2$ . When  $p = 2 \bmod 3$  we only need to consider the first of these three presentations, and the automorphism group has order  $(p^3 - 1)p^{12}$ ,  $(p - 1)(p^2 - 1)p^{12}$ ,  $(p - 1)^2 p^{13}$ ,  $(p - 1)^3 p^{12}$  depending on whether  $x^3 - \lambda x + 1$  has 0, 1, 2 or 3 roots. When  $p = 1 \bmod 3$  and  $\lambda \neq 0$  then the automorphism groups have the same orders as in the case  $p = 2 \bmod 3$ , but now depending on the numbers of roots of  $x^3 - \lambda x + 1$



for the first presentation, the number of roots of  $x^3 - \lambda x + \omega$  for the second presentation, and depending on the number of roots of  $x^3 - \lambda x + \omega^2$  for the third presentation. When  $x = 0$  and  $p = 1 \pmod{3}$  then the automorphism groups have orders  $3(p-1)^2(p^2-1)p^{12}$ ,  $3(p^3-1)p^{12}$ ,  $3(p^3-1)p^{12}$ .

There are three orbits of subspaces of  $M$  of dimension 9 which intersect  $N$  in the last of these subspaces.

So the total number of orbits of non-allowable subalgebras of  $M$  of dimension 9 is  $3p + 24 + \gcd(p-1, 3)$ . The number of immediate descendants of dimension 9 can be obtained by subtracting away this number from the total number of orbits of subspaces of  $M$  dimension 9, and so is

$$p^6 + 2p^5 + 6p^4 + 11p^3 + 27p^2 + 74p + 138 + (p^2 + \frac{11}{2}p + \frac{25}{2}) \gcd(p-1, 3) \\ + (p^2 + \frac{1}{2}p + \frac{3}{2}) \gcd(p+1, 3) + (p+6) \gcd(p-1, 4) + \gcd(p-1, 5).$$

### Group 7.4.3

$$\langle a, b, c, d \mid [d, a], [c, b], [d, c] \rangle$$

The number of conjugacy classes is  $3p^4 - 3p^2 + p$ , and the automorphism group has order  $2(p-1)^4p^{16}$ . The nucleus has dimension 8, and is spanned by  $baa$ ,  $bab$ ,  $bac$ ,  $bad$ ,  $caa$ ,  $cac$ ,  $dbb$ ,  $dbd$ . The number of orbits of subspaces of dimension 6 in the nucleus under the automorphism group is

$$\frac{1}{2}(p^5 + 4p^4 + 15p^3 + 48p^2 + 139p + 477 + \gcd(p-1, 3)(4p + 26) \\ + \gcd(p-1, 4)(3p + 15) + 2 \gcd(p-1, 5)).$$

The multiplier  $M$  has dimension 11, and the number of orbits of subspaces of  $M$  of dimension 9 under the automorphism group is

$$\frac{1}{2}(p^5 + 4p^4 + 21p^3 + 85p^2 + 355p + 1875) \\ + \frac{1}{2}(p^2 + 13p + 103) \gcd(p-1, 3) \\ + \frac{1}{2}(3p + 44) \gcd(p-1, 4) \\ + \gcd(p-1, 3) \gcd(p-1, 4) + 3 \gcd(p-1, 5).$$

The number of descendants of dimension 9 is the number of orbits of allowable subspaces of  $M$  of dimension 9. So to we need to find the number of orbits of non-allowable subspaces of  $M$  of dimension 9. A non-allowable subspace of  $M$  of dimension 9 must either contain the nucleus, or intersects the nucleus in

a subspace of dimension 7. To compute these subspaces we adopt a different presentation for  $L$ .

$$\langle a, b, c, d \mid ca, da, db, \text{ class } 2 \rangle.$$

With this presentation the nucleus of dimension 8 is spanned by  $baa$ ,  $bab$ ,  $bac$ ,  $cbb$ ,  $cbc$ ,  $cbd$ ,  $dcc$ ,  $dcd$ .

The automorphism group is determined by its action on the Frattini quotient of the group and is given by matrices in  $\text{GL}(4, p)$  of the following two forms:

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & \gamma & 0 & \delta \\ \varepsilon & 0 & \zeta & \eta \\ 0 & 0 & 0 & \theta \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & \alpha \\ \beta & 0 & \gamma & \delta \\ \varepsilon & \zeta & 0 & \eta \\ \theta & 0 & 0 & 0 \end{pmatrix}.$$

We consider the action of these automorphisms on the dual space of the nucleus. It turns out that there are  $p+26$  orbits of one dimensional subspaces of this dual space. A set of basis elements for representatives of these orbits is as follows:

$$\begin{aligned} &(0, 0, 0, 0, 0, 0, 0, 1), \\ &(0, 0, 0, 0, 0, 0, 1, 0), \\ &(0, 0, 0, 0, 0, 1, 0, 0), \\ &(0, 0, 0, 0, 0, 1, 0, 1), \\ &(0, 0, 0, 0, 0, 1, 1, 0), \\ &(0, 0, 0, 0, 0, 1, 1, 1), \\ &(0, 0, 0, 0, 1, 0, 0, 0), \\ &(0, 0, 0, 0, 1, 0, 0, 1), \\ &(0, 0, 0, 1, 0, 0, 0, k), \end{aligned}$$

where  $k$  is not a square,

$$\begin{aligned} &(0, 0, 0, 1, 0, 0, 1, 0), \\ &(0, 0, 0, 1, 0, 1, 1, k), \end{aligned}$$

where  $1 + k$  is not a square,

$$\begin{aligned}
&(0, 0, 0, 1, 1, 0, 0, 0), \\
&(0, 0, 1, 0, 0, 0, 0, 1), \\
&(0, 0, 1, 0, 0, 0, 1, 0), \\
&(0, 0, 1, 0, 0, 1, 0, 0), \\
&(0, 0, 1, 0, 0, 1, 1, 0), \\
&(0, 1, 0, 0, 0, 0, 0, 1), \\
&(0, 1, 0, 0, 0, 0, 1, 0), \\
&(0, 1, 0, 0, 0, 1, 1, 0), \\
&(0, 1, 0, 0, 0, 1, 1, 1), \\
&(0, 1, 1, 0, 0, 0, 0, 1), \\
&(0, 1, 1, 0, 0, 1, \lambda, 0),
\end{aligned}$$

with  $1 \leq \lambda \leq p - 1$ , and

$$\begin{aligned}
&(1, 0, 0, 0, 0, 0, 0, 1), \\
&(1, 0, 0, 0, 0, 1, 0, 1), \\
&(1, 0, 0, 0, 0, 1, 1, -1), \\
&(1, 0, 0, 0, 0, 1, 1, -k), \\
&(1, 0, 0, 0, k, 0, 0, 1),
\end{aligned}$$

where in the last two  $k$  is not a square, and finally

$$(1, 0, 0, 1, 0, 1, 1, k)$$

where neither  $1 + k$  nor  $-k$  are squares. The annihilators of these elements of the dual space of  $N$  give subspaces of  $N$  of dimension 7.

There are 3 orbits of subspaces of  $M$  of dimension 9 which contain the nucleus. For the first 21 of the subspaces  $S$  giving representatives for the  $p + 26$  orbits of subspaces of the nucleus  $N$  of dimension 7, the number of orbits of subspaces of  $M$  of dimension 9 which contain subspaces intersecting  $N$  in  $S$  is as follows: 7, 9, 7, 7, 5,  $\frac{p+7}{2}$ , 9, 5, 7, 7,  $\frac{p+7}{2}$ , 5, 9, 8, 3,  $p + 2$ , 7,  $4 + \gcd(p - 1, 3)$ ,  $p + 2$ ,  $4 + \gcd(p - 1, 5)$ ,  $p + 6$ . For the subspaces corresponding to  $(0, 1, 1, 0, 0, 1, \lambda, 0)$ , the relevant number of orbits is  $\frac{p+3}{2}$  if  $-\lambda$  is not a square in  $\text{GF}(p)$ ,  $\frac{p+5}{2}$  if  $-\lambda$  is a square in  $\text{GF}(p)$  and  $\lambda \neq -4$ , and  $\frac{p+7}{2}$  if  $\lambda = -4$ . For the last six subspaces the relevant numbers of orbits are

$$4 + \gcd(p - 1, 3),$$

$$\begin{aligned}
& \frac{1}{2}(p + 7 + 2 \gcd(p - 1, 3) + \gcd(p - 1, 4)), \\
& \frac{1}{8}(p^2 + 6p + 9 + 2 \gcd(p - 1, 4)), \\
& \frac{p^2 - 1}{4} + p + 2, \\
& \frac{p - 1}{2} + 4 + \frac{1}{2} \gcd(p - 1, 4) + \gcd(p - 1, 3), \\
& \frac{1}{8}(p^2 + 6p + 9 + 2 \gcd(p - 1, 4)).
\end{aligned}$$

So the total number of non-allowable subspaces of  $M$  of dimension 9 is

$$p^2 + 9p + 137 + 4 \gcd(p - 1, 3) + \frac{3}{2} \gcd(p - 1, 4) + \gcd(p - 1, 5).$$

Subtracting this total from the number of orbits of subspaces of  $M$  of dimension 9 we obtain the number of descendants of dimension 9:

$$\begin{aligned}
& \frac{1}{2}(p^5 + 4p^4 + 21p^3 + 83p^2 + 337p + 1601) \\
& + \frac{1}{2}(p^2 + 13p + 95) \gcd(p - 1, 3) \\
& + \frac{1}{2}(3p + 41) \gcd(p - 1, 4) \\
& + \gcd(p - 1, 3) \gcd(p - 1, 4) + 2 \gcd(p - 1, 5).
\end{aligned}$$

#### Group 7.4.4

$$\langle a, b, c, d \mid [d, a] = [c, b], [d, b], [d, c] \rangle$$

The number of conjugacy classes is  $2p^4 + p^3 - 2p^2$ , and the automorphism group has order  $(p - 1)(p^2 - 1)(p^2 - p)p^{17}$ . The nucleus has dimension 8, and is spanned by  $baa$ ,  $bab$ ,  $bac$ ,  $bad$ ,  $caa$ ,  $cab$ ,  $cac$ ,  $cad$ . The number of orbits of subspaces of dimension 6 in the nucleus under the automorphism group is

$$p^3 + 3p^2 + 25p + 64 + \gcd(p - 1, 3) + 2 \gcd(p - 1, 4).$$

The multiplier has dimension 11, and the number of orbits of subspaces of the multiplier of dimension 9 is

$$p^3 + 5p^2 + 52p + 222 + 2 \gcd(p - 1, 3) + 3 \gcd(p - 1, 4)$$

when  $p > 5$ , and 748 when  $p = 5$ . Not all these subspaces are allowable. A subspace which is not allowable must either contain the nucleus, or must intersect the nucleus in a subspace of dimension 7. There are 2 orbits of 9 dimensional subspaces of the multiplier which contain the nucleus. There are  $p + 10$  orbits of subspaces of dimension 7 in the nucleus. Representatives for these subspaces are given by the following one dimensional subspaces of the dual space of the nucleus:

$$\begin{aligned}
& \langle (0, 0, 0, 0, 0, 0, 0, 1) \rangle, \\
& \langle (0, 0, 0, 0, 0, 0, 1, 0) \rangle, \\
& \langle (0, 0, 0, 0, 0, 1, 0, 0) \rangle, \\
& \langle (0, 0, 0, 0, 1, 0, 0, 0) \rangle, \\
& \langle (0, 0, 0, 1, 0, 0, 1, 0) \rangle, \\
& \langle (0, 0, 0, 1, 0, 1, 0, 0) \rangle, \\
& \langle (0, 0, 0, 1, -\omega, 0, 0, 0) \rangle, \\
& \langle (0, 0, 1, 0, 0, x, 0, 0) \rangle \ x \neq 0, \ x \sim x^{-1}, \\
& \langle (0, 0, 1, 0, 0, -1, 1, 0) \rangle, \\
& \langle (0, 1, 0, 0, 0, 0, -\omega, 0) \rangle, \\
& \langle (0, 1, 0, 0, 0, 1, x, 0) \rangle \text{ all } x \text{ with } 1 - 4x \text{ not square,} \\
& \langle (0, 1, 0, 0, 1, 0, 0, 0) \rangle.
\end{aligned}$$

(Here  $x \sim x^{-1}$  indicates that  $x$  and  $x^{-1}$  give subspaces in the same orbit.) For the first seven of these subspaces  $S$ , there are (respectively) 4, 5, 6, 7, 5,  $\frac{p+5}{2}$ ,  $\frac{p+5}{2}$  orbits of subspaces of  $M$  of dimension 9 which contain representatives intersecting  $N$  in  $S$ . The next subspace  $S$  is indexed by a parameter  $x$ , which runs through a set of representatives for equivalence classes of non-zero elements of  $\text{GF}(p)$  under the equivalence relation  $x \sim x^{-1}$ . When  $x = 1$  we have 3 orbits of subspaces of  $M$  of dimension 9 which contain representatives intersecting  $N$  in  $S$ . When  $x = -1$  we have 3 orbits when  $p = 5$  and 2 orbits when  $p > 5$ . When  $x \neq \pm 1$  we have 4 orbits, except (for  $p > 5$ ) when  $x = \frac{2}{3}$  or  $\frac{3}{2}$  we have 5 orbits. The next subspace gives 4 orbits if  $p = 5$  and 3 orbits if  $p > 5$ . The next subspace gives 2 orbits. The penultimate subspace is indexed by  $x$ , where  $x$  takes on the  $\frac{p-1}{2}$  values in  $\text{GF}(p)$  such that  $1 - 4x$  is not square. Each of the corresponding subspaces gives 2 orbits. The last subspace gives 5 orbits.

So when  $p = 5$  there are 64 orbits of non-allowable subspaces of  $M$  of dimension 9, and when  $p > 5$  there are  $4p + 43$  orbits. By subtracting these numbers from the grand total of all orbits of 9 dimensional subspaces of  $M$ ,

we obtain the number of descendants of dimension 9. When  $p = 5$  there are 684 descendants, and when  $p > 5$  the number of descendants is

$$p^3 + 5p^2 + 48p + 179 + 2 \gcd(p - 1, 3) + 3 \gcd(p - 1, 4).$$

### Group 7.4.5

$$\langle a, b, c, d \mid [c, a], [d, a] = [c, b], [d, b] \rangle$$

The number of conjugacy classes is  $2p^4 + p^3 - 2p^2$ , and the automorphism group has order  $(p^2 - 1)^2(p^2 - p)p^{13}$ . The nucleus has dimension 8 and is spanned by  $baa, bab, bac, bad, cbc, cbd, dcc, dcd$ . The number of orbits of subspaces of dimension 6 in the nucleus under the automorphism group is

$$p^6 + p^5 + 5p^4 + 6p^3 + 18p^2 + 22p + 51 + (4p + 11) \gcd(p - 1, 3) \\ + 2 \gcd(p - 1, 4) + \gcd(p - 1, 5).$$

The multiplier  $M$  has dimension 11, and the number of orbits under the automorphism group of subspaces of  $M$  of dimension 9 is

$$p^6 + p^5 + 6p^4 + 13p^3 + 34p^2 + 72p + 205 + (p^2 + 6p + 39) \gcd(p - 1, 3) \\ + 4 \gcd(p - 1, 4) + \gcd(p - 1, 5) + \gcd(p - 1, 9).$$

The number of descendants of dimension 9 is the number of orbits of allowable subspaces of  $M$  of dimension 9. So to compute the number of descendants of dimension 9 we need to compute the number of orbits of non-allowable subspaces of  $M$  of dimension 9. A non-allowable subspace of  $M$  of dimension 9 must either contain the nucleus, or intersect the nucleus in a subspace of dimension 7. To compute these subspaces we adopt a different presentation for  $L$ .

$$\langle a, b, c, d \mid ba, da - cb, dc, \text{class } 2 \rangle.$$

With this presentation, nucleus of dimension 8, spanned by  $caa, cab, cac, cad, cbb, cbd, dbb, dbd$ . The action of the automorphism group on  $L/L^2$  is given by matrices in  $\text{GL}(4, p)$  of the form

$$\begin{pmatrix} \lambda\alpha & \lambda\beta & \mu\alpha & \mu\beta \\ \lambda\gamma & \lambda\delta & \mu\gamma & \mu\delta \\ \nu\alpha & \nu\beta & \xi\alpha & \xi\beta \\ \nu\gamma & \nu\delta & \xi\gamma & \xi\delta \end{pmatrix},$$

with  $(\alpha\delta - \beta\gamma)(\lambda\xi - \mu\nu) \neq 0$

As usual we compute the orbits of one dimensional subspaces of the dual space of the nucleus under the action of the automorphism group. It turns out that there are  $2p + 10 + \gcd(p - 1, 3)$  orbits of these subspaces. Bases for representative subspaces are as follows:

$$\begin{aligned} &(0, 0, 0, 0, 0, 0, 0, 1), \\ &(0, 0, 0, 0, 0, 0, 1, 0, 0), \\ &(0, 0, 0, 0, 0, 0, 1, 1, 0), \\ &(0, 0, 0, 1, 0, 0, 0, 0, 1), \\ &(0, 0, 0, 1, 0, 0, 0, 0, \omega), \\ &(0, 0, 0, 1, 0, 0, 1, 0, 0), \\ &(0, 0, 0, 1, 1, 0, 0, 0, 0), \\ &(0, 0, 0, 1, 1, 0, 0, 1, 0), \\ &(0, 0, 0, 1, 1, 0, 0, 0, \omega), \end{aligned}$$

$$(0, 0, 1, 0, 0, 0, 0, 0, \omega) \text{ if } p = 1 \bmod 3,$$

$$(0, 0, 1, 0, 0, 0, 1, 0, 0),$$

$$(0, 0, 1, 0, 0, 1, 0, 0, \lambda) \text{ if } p = 2 \bmod 3 \text{ for any one } \lambda \text{ such that } x^3 + 3x + \lambda \text{ is irreducible,}$$

$$\begin{aligned} &(0, 0, 1, 0, 0, 1, 1, 0, 0), \\ &(0, 0, 1, 0, 0, 0, \omega, 1, 0), \\ &(0, 0, 1, 0, 1, 0, 0, 0, 2), \end{aligned}$$

$$\begin{aligned} &(0, 0, 1, 0, 1, 0, 0, 0, 2\omega) \text{ if } p = 1 \bmod 3, \\ &(0, 0, 1, 0, 1, 0, 0, 0, 2\omega^2) \text{ if } p = 1 \bmod 3, \end{aligned}$$

$$(0, 0, 1, 0, 1, 0, 1, 0, \lambda) \text{ for all } \lambda \neq 0, \frac{1}{27},$$

$$(0, 1, 1, 0, 0, 0, \lambda, \mu, \nu), (p - 1 \text{ of them}).$$

For each of these elements  $f$  of the dual space of the nucleus we take a set of basis elements of the annihilator of  $f$  as relators.

There are 3 orbits of subspaces of  $M$  of dimension 9 which contain the nucleus. For the first 14 of the  $2p + 10 + \gcd(p - 1, 3)$  subspaces  $S$  giving representatives for the  $2p + 10 + \gcd(p - 1, 3)$  orbits of subspaces of the nucleus  $N$  of dimension 7, the number of orbits of subspaces of  $M$  of dimension 9 which contain subspaces intersecting  $N$  in  $S$  is as follows: 7, 7, 5, 7, 7,  $2p + 5 + \gcd(p - 1, 4)$ ,  $p + 4$ ,  $\frac{1}{2}(p^2 + 2p + 3)$ ,  $\frac{1}{2}(p^2 + 2p + 3)$ , 7,  $\frac{1}{2}(p + 3)(\gcd(p - 1, 3) + 1) + 1$ , 7,  $\frac{1}{2}(p^2 + 2p + 3)$ ,  $\frac{1}{2}(p^2 + 2p + 3)$ . For the next subspace the relevant number of orbits is  $\frac{1}{2}(p^2 + 2p + 3)$  when  $p \equiv 2 \pmod{3}$ , and  $\frac{1}{12}(p^2 + 4p + 31)$  when  $p \equiv 1 \pmod{3}$ . The next two subspaces only arise when  $p \equiv 1 \pmod{3}$ , and then the relevant number of orbits is  $\frac{1}{3}(p^2 + p + 7)$  in both cases. The next subspace is actually a family of  $p - 2$  subspaces indexed by a parameter  $\lambda$ . It turns out that the relevant number of orbits depends on the number of roots over  $\text{GF}(p)$  of the polynomial  $2x^3 + x^2 - \lambda$ . The discriminant of this polynomial is  $\frac{1}{4}\lambda(1 - 27\lambda)$ , and the discriminant is zero if  $\lambda = 0$  or  $\frac{1}{27}$ , the two excluded values of  $\lambda$ . So the number of roots for  $\lambda \neq 0, \frac{1}{27}$  is 0, 1 or 3. If there are  $n$  roots, then the relevant number of orbits is then

$$\frac{p^2 + p + 1 + n(p + 2)}{n + 1}.$$

Intriguingly, if we pick an integer  $\lambda$  such that the Galois group of  $2x^3 + x^2 - \lambda$  over the rationals is  $S_3$ , then the number of roots of  $2x^3 + x^2 - \lambda$  over  $\text{GF}(p)$  is not PORC. But as  $\lambda$  ranges over  $\text{GF}(p)$  the number of times 0, 1, 2, or 3 roots arises is PORC. The following table gives the number of times  $n$  roots arises ( $n = 0, 1, 2, 3$ ) as  $\lambda$  ranges over  $\text{GF}(p)$ .

$n$	0	1	2	3
$p \equiv 1 \pmod{3}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	2	$\frac{p-7}{6}$
$p \equiv 2 \pmod{3}$	$\frac{p+1}{3}$	$\frac{p-3}{2}$	2	$\frac{p-5}{6}$

We obtain the figures for  $n = 0$  by calculating the number of irreducible polynomials of the form  $2x^3 + x^2 - \lambda$ , and the figures for  $n = 1$  by calculating the number of values of  $\lambda$  for which the discriminant is not a square. We get 2 roots if  $\lambda = 0$  or  $\frac{1}{27}$ , and the figure for  $n = 3$  then follows from the fact that the rows have to add up to  $p$ . So the sum of all the relevant number of orbits over these subspaces is PORC.

Finally we have  $p - 1$  subspaces corresponding to elements in the dual space of the nucleus of the form  $(0, 1, 1, 0, 0, \lambda, \mu, \nu)$  with  $\lambda, \mu \neq 0$ . For  $\frac{p-1}{2}$  of these subspaces the relevant number of orbits is  $\frac{1}{2}(p^2 + 2p + 3)$ , and for



$\frac{p-5}{2}$  of the subspaces the relevant number of orbits is  $\frac{1}{4}(p^2 + 4p + 7)$ . Finally we have the subspaces corresponding to

$$(0, 1, 1, 0, 0, \omega, \omega, 0) \text{ and } (0, 1, 1, 0, 0, -3\omega, -27\omega, 0).$$

For the first of these the relevant number of orbits is  $p + 2$  when  $p = 1 \pmod 3$  and  $2p + 3$  when  $p = 2 \pmod 3$ , and for the second the relevant number of orbits is  $p + 2$ .

So the total number of non-allowable subspaces of  $M$  of dimension 9 is

$$p^3 + 3p^2 + 10p + 56 + 4 \gcd(p - 1, 3) + \gcd(p - 1, 4)$$

We obtain the number of descendants of dimension 9 by subtracting this number from the total number of subspaces of  $M$  of dimension 9:

$$\begin{aligned} & p^6 + p^5 + 6p^4 + 12p^3 + 31p^2 + 62p + 149 + (p^2 + 6p + 35) \gcd(p - 1, 3) \\ & + 3 \gcd(p - 1, 4) + \gcd(p - 1, 5) + \gcd(p - 1, 9). \end{aligned}$$

#### Group 7.4.6

$$\langle a, b, c, d \mid [d, b] = [c, a]^\omega, [d, c] = [b, a], [c, b] \rangle$$

The number of conjugacy classes is  $p^4 + 2p^3 - p^2 - p$ , and the automorphism group has order  $2(p^2 - 1)^2 p^{16}$ . The nucleus has dimension 8 and is spanned by  $baa, bac, bad, caa, cac, cad, daa, dad$ . The number of orbits of subspaces of dimension 6 in the nucleus under the automorphism group is

$$\frac{1}{2}(p^5 + 5p^3 + 8p^2 + 23p + 29 - 2p \gcd(p - 1, 3) + (p + 1) \gcd(p - 1, 4)).$$

The  $p$ -multiplier  $M$  has dimension 11, and the number of orbits of subspaces of codimension 2 in  $M$  under the automorphism group is

$$\frac{1}{2}(p^5 + 5p^3 + 17p^2 + 45p + 137 - (p^2 + 3p + 13) \gcd(p - 1, 3) + (p + 4) \gcd(p - 1, 4)).$$

Not all subspaces of  $M$  of dimension 9 are allowable, and to obtain the total number of descendants of 7.4.6 of order  $p^9$  with exponent  $p$  we need to subtract the number of orbits of non-allowable subspaces of dimension 9 from the total number of orbits of subspaces of dimension 9. A non-allowable subspace of dimension 9 either contains the nucleus, or intersects the nucleus in a subspace of dimension 7. There are 2 orbits of subspaces of  $M$  of dimension 9 containing the nucleus.

To compute the number of orbits of subspaces of  $M$  of dimension 9 which intersect the nucleus in a subspace of dimension 7, we switch to an alternative presentation for group 7.4.6:

$$\langle a, b, c, d \mid da, db - \omega ca, dc - ba, \text{class } 2 \rangle.$$

With this presentation the nucleus of dimension 8, spanned by  $baa, bab, bac, caa, cab, cac, cbb, cbc$ .

The action of the automorphism group on  $L/L^2$  is given by the following matrices in  $\text{GL}(4, p)$ :

$$\begin{pmatrix} \alpha & 0 & 0 & \beta \\ \lambda & \delta & \omega\gamma & \mu \\ \nu & \pm\gamma & \pm\delta & \xi \\ \pm\omega\beta & 0 & 0 & \pm\alpha \end{pmatrix}.$$

As usual we compute the orbits of one dimensional subspaces of the dual space of the nucleus under the action of the automorphism group. It turns out that there are  $p + 5$  orbits of these subspaces. Bases for representative subspaces are as follows:

$$\begin{aligned} &(0, 0, 0, 0, 0, 0, 0, 1), \\ &(0, 0, 0, 0, 0, 1, 0, 0), \\ &(0, 0, 0, 1, 0, 0, 0, 0), \\ &(0, 0, 0, 1, 0, 0, 0, 1), \end{aligned}$$

$$(0, 0, 0, 1, 0, 0, 1, \lambda) \text{ for one } \lambda \text{ such that } \omega^2\lambda^2 - \omega \text{ is not a square,}$$

and  $p$  elements

$$(0, 0, 1, 0, \lambda, \mu, 0, 0)$$

with  $\lambda \neq 0$ . The numbers of orbits of subspaces of  $M$  of dimension 9 corresponding to the first five of these are 3,

$$\begin{aligned} &3, \\ &\frac{1}{2}(p + 5), \\ &6 - \gcd(p - 1, 3), \\ &\frac{1}{4}(p^2 + (6 - \gcd(p - 1, 4))p + 5 + \gcd(p - 1, 4)), \\ &\frac{1}{4}(p^2 + \gcd(p - 1, 4)p + 3 + \gcd(p - 1, 4)). \end{aligned}$$

In the last case, the number of orbits of subspaces of  $M$  of dimension 9 corresponding to the  $p$  representative subspaces are as follows. We have three representative subspaces giving  $6 - \gcd(p-1, 3)$ , 2 and  $\frac{p+7}{2}$  orbits respectively. We have  $\frac{p-1}{2}$  representative subspaces each giving  $\frac{p+3}{2}$  orbits. And finally we have  $\frac{p-5}{2}$  representative subspaces each giving  $\frac{p+5}{2}$  orbits.

So the total number of orbits of non-allowable subspaces of  $M$  of dimension 9 is

$$p^2 + 3p + 20 - 2 \gcd(p-1, 3) + \frac{1}{2} \gcd(p-1, 4).$$

We obtain the number of descendants of dimension 9 by subtracting this number from the total number of orbits of subspaces of  $M$  of dimension 9, giving:

$$\frac{1}{2}(p^5 + 5p^3 + 15p^2 + 39p + 97 - (p^2 + 3p + 9) \gcd(p-1, 3) + (p+3) \gcd(p-1, 4)).$$

## 7.5 Five generator groups

### Group 7.5.1

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $2p^5 - p^3$  and the order of the automorphism group is  $(p-1)(p^2-1)^2(p^2-p)^2p^{18}$ . The nucleus has dimension 5, and is generated by  $baa$ ,  $bab$ ,  $bac$ ,  $caa$ ,  $cac$ . From the calculation of the descendants of  $\langle a, b, c \mid [c, b], \text{class 2, exponent } p \rangle$ , we may assume that one of the following holds:

$$\begin{aligned} caa &= cab = cac = 0, \\ bab &= bac = cab = cac = 0, \\ bac &= cac = 0, caa = bab, \\ baa &= bac = cac = 0, \\ bac &= caa = 0, cac = bab, \\ bac &= caa = 0, cac = \omega bab, \\ \\ bac &= 0, caa = baa, cac = -bab, \\ baa &= bac = caa = 0, \\ bac &= caa = 0, baa = cac, \end{aligned}$$

$$\begin{aligned}
& bac = 0, baa = cac, caa = bab, \\
& bac = 0, baa = cac, caa = \omega bab, (p = 1 \bmod 3) \\
& \quad baa = caa = cac = 0, \\
& \quad baa = cac = 0, caa = bab, \\
& \quad caa = cac = 0, baa = bab, \\
& \quad baa = caa = 0, cac = \omega bab, \\
& \quad baa = 0, caa = bac, cac = \omega bab, \\
& baa = 0, caa = kbab + bac, cac = \omega bab, (p = 2 \bmod 3),
\end{aligned}$$

where  $k$  is any element of  $\mathbb{Z}_p$  which is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)}.$$

The number of descendants of class 3 with order  $p^9$  and exponent  $p$  in each of the first 9 cases above is

$$\begin{aligned}
& 2p + 60 + 2 \gcd(p - 1, 3), \\
& 4p + 27, \\
& 3p + 25, \\
& p + 61 + \gcd(p - 1, 4), \\
& \frac{1}{2}(p^2 + 11p + 56 + 3 \gcd(p - 1, 4))/2, \\
& \frac{1}{2}(p^2 + 11p + 56 + 3 \gcd(p - 1, 4))/2, \\
& p + 49, \\
& \frac{1}{2}(p^2 + 6p + 67 + 9 \gcd(p - 1, 3) + \gcd(p - 1, 5), \\
& p^3 + 3p^2 + 9p + 27 + 3 \gcd(p - 1, 3) + 3 \gcd(p - 1, 4).
\end{aligned}$$

When  $p = 1 \bmod 3$  then the number of descendants of class 3 with order  $p^9$  and exponent  $p$  in cases 10 and 11 above combined is

$$\frac{1}{2}(p^4 + 2p^3 + 4p^2 + 14p + 39)$$

and when  $p = 2 \bmod 3$  the number is

$$\frac{1}{2}(p^4 + 2p^3 + 4p^2 + 8p + 15).$$

The number of descendants of class 3 with order  $p^9$  and exponent  $p$  in each of cases 12, 13, 14, 15 above is

$$\begin{aligned} & 2p + 35, \\ & p^3 + 3p^2 + 7p + 27 + 3 \gcd(p - 1, 3), \\ & p^2 + 7p + 21, \\ & \frac{1}{2}(p^2 + 4p + 33 - \gcd(p - 1, 3) + \gcd(p + 1, 5)). \end{aligned}$$

Finally, in cases 16 and 17 above combined, the number of descendants is

$$\frac{1}{2}(p^4 + 2p^3 + 4p^2 + 8p + 15)$$

when  $p \equiv 1 \pmod{3}$ , and

$$\frac{1}{2}(p^4 + 2p^3 + 4p^2 + 10p + 23)$$

when  $p \equiv 2 \pmod{3}$ .

### Group 7.5.2

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [b, a], [d, c], [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - 2p^2 + p$  and the order of the automorphism group is  $2(p - 1)(p^2 - 1)^2(p^2 - p)^2p^{14}$ . The nucleus has dimension 4. It is convenient to take another presentation for this group:

$$\langle a, b, c, d, e \mid [c, a], [c, b], [d, a], [d, b], [e, a], [e, b], [e, c], [e, d] \rangle.$$

The nucleus has dimension 4 and is spanned by  $baa$ ,  $bab$ ,  $dcc$ ,  $dcd$ . We need to factor out a two dimensional subspace of the nucleus, and we can assume that one of the following pairs of relations holds:

$$\begin{aligned} dcc &= dcd = 0, \\ bab &= dcd = 0, \\ bab - dcc &= dcd = 0, \\ dcc - baa &= dcd - bab = 0, \end{aligned}$$

or, when  $p = 1 \bmod 3$ ,

$$dcc - baa = dcd - \omega bab = 0.$$

The number of descendants of order  $p^9$  with class 3 and exponent  $p$  in the first 3 cases is

$$\begin{aligned} & p + 65, \\ & \frac{1}{2}(4p + 135 + 5 \gcd(p - 1, 3)), \\ & 6p + 93 + 5 \gcd(p - 1, 3) + 4 \gcd(p - 1, 4), \end{aligned}$$

and the number of descendants in the last two cases combined is

$$\frac{1}{2}(2p^2 + 13p + 11 + 30 \gcd(p - 1, 3) + \gcd(p - 1, 4)).$$

### Group 7.5.3

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p^2$  and the order of the automorphism group is  $(p-1)^2(p^2-1)(p^2-p)p^{18}$ . The nucleus has dimension 4, and is spanned by  $baa$ ,  $bab$ ,  $bac$ ,  $bad$ . We may assume that one of the following four sets of relations holds:

$$\begin{aligned} bac &= bad = 0, \\ baa &= bac = 0, \\ bab &= bac = 0, \\ baa &= bab = 0. \end{aligned}$$

The number of descendants of order  $p^9$  with class 3 and exponent  $p$  in these four cases are:

$$\begin{aligned} & 4p + 48 + \gcd(p - 1, 3), \\ & 3p + 69, \\ & 5p + 46, \\ & 2p + 44. \end{aligned}$$

### Group 7.5.4

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega, [e, a], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + p^3 - p$  and the order of the automorphism group is  $2(p-1)(p^4-1)(p^4-p^2)p^{14}$ . The nucleus has dimension 4, and is spanned by  $baa$ ,  $bab$ ,  $bac$ ,  $bad$ . We may assume that one of the following three sets of relations holds:

$$\begin{aligned} bac &= bad = 0, \\ baa &= bad = 0, \end{aligned}$$

or when  $p \equiv 2 \pmod{3}$

$$bac = bad - xbaa = 0,$$

where  $x$  is chosen so that  $y^3 - y - \frac{4x}{3(x^2+3)}$  is irreducible. Note that  $x^2 + 3$  can never be zero since  $p \equiv 2 \pmod{3}$ . Also note that  $x$  cannot be zero. We also require that  $x^2 \neq 3$ .

The number of descendants of order  $p^9$  with class 3 and exponent  $p$  in the first and last cases combined is

$$\frac{1}{2}(2p^2 + 13p + 65 - 8 \gcd(p-1, 3) + \gcd(p-1, 4)).$$

The number of descendants in the second case is

$$\frac{1}{2}(4p + 35 - 3 \gcd(p-1, 3)).$$

### Group 7.5.5

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], [e, d] = [b, a] \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p^2$  and the order of the automorphism group is  $(p-1)^2(p^2-1)(p^2-p)p^{15}$ . The nucleus has dimension 2. The number of descendants of order  $p^9$  with exponent  $p$  is

$$p^3 + 4p^2 + 16p + 4 \gcd(p-1, 4) + 6 \gcd(p-1, 3) + 80.$$

### Group 7.5.6

$$\langle a, b, c, d, e \mid [c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], [e, d] = [b, a] \rangle$$

The number of conjugacy classes is  $p^5 + p^3 - p$  and the order of the automorphism group is  $(p-1)(p^2-1)(p^2-p)p^{14}$ . This group has no class 3 descendants of order  $p^9$ .

## 7.6 Six generator groups

### Group 7.6.1

$$\langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle \times \langle f \rangle$$

The number of conjugacy classes is  $p^6 + p^5 - p^4$ , and the automorphism group has order  $(p^7 - p^5)(p^7 - p^6)(p^5 - p)(p^5 - p^2)(p^5 - p^3)(p^5 - p^4)$ . The nucleus has dimension 2. If  $G$  is an immediate descendant of 7.6.1 of order  $p^9$  with exponent  $p$ , then the subgroup of  $G$  generated by  $a, b$  must be the free class 3 group with exponent  $p$  of rank 2, and has order  $p^5$ . The subgroup of  $G$  generated by  $c, d, e, f$  is either elementary abelian, or is one of 5.4.1, 5.4.2, 6.4.1, 6.4.2, 6.4.3, 6.4.4. The number of descendants in the seven cases is 8, 15, 2, 15, 3, 3, 2, making 48 descendants in all.

### Group 7.6.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle \times \langle e \rangle \times \langle f \rangle$$

The number of conjugacy classes is  $p^6 + p^3 - p^2$ , and the automorphism group has order  $(p^7 - p^3)(p^7 - p^6)(p^5 - p^3)p^4(p^3 - p)(p^3 - p^2)$ . This group is terminal.

### Group 7.6.3

$$\langle a, b \rangle \times_{[b,a]=[d,c]=[f,e]} \langle c, d \rangle \times_{[b,a]=[d,c]=[f,e]} \langle e, f \rangle$$

The number of conjugacy classes is  $p^6 + p - 1$ , and the automorphism group has order  $(p^7 - p)(p^7 - p^6)(p^5 - p)p^4(p^3 - p)p^2$ . This group is terminal.

## 8 Order $p^8$

### 8.4 Four generator groups

#### Group 8.4.1

$$\langle a, b, c, d \mid [b, a], [c, a] \rangle$$

The number of conjugacy classes is  $2p^5 + p^4 - 2p^3$ , and the automorphism group has order  $(p - 1)^2(p^2 - 1)(p^2 - p)p^{21}$ . The nucleus has dimension 12. There are  $9p + 58 + 3 \gcd(p - 1, 3)$  orbits of subspaces of the nucleus of codimension 1, and a total of  $9p + 71 + 3 \gcd(p - 1, 3)$  descendants of order  $p^9$  with exponent  $p$ .



### Group 8.4.2

$$\langle a, b, c, d \mid [b, a], [d, c] \rangle$$

The number of conjugacy classes is  $p^5 + 3p^4 - 2p^3 - 2p^2 + p$ , and the automorphism group has order  $2(p^2 - 1)^2(p^2 - p)^2p^{16}$ . The nucleus has dimension 12. The number of orbits of subspaces of the nucleus of codimension 1 is

$$\begin{aligned} & \frac{1}{2}(p^4 + 2p^3 + 7p^2 + 19p + 2 \gcd(p - 1, 3) + 2 \gcd(p - 1, 4) \\ & + \frac{1}{2} \gcd(p - 1, 5) + \frac{1}{2} \gcd(p + 1, 5) + 76). \end{aligned}$$

The total number of descendants of order  $p^9$  with exponent  $p$  is

$$\begin{aligned} & \frac{1}{2}(p^4 + 2p^3 + 7p^2 + 19p + 2 \gcd(p - 1, 3) + 2 \gcd(p - 1, 4) \\ & + \frac{1}{2} \gcd(p - 1, 5) + \frac{1}{2} \gcd(p + 1, 5) + 88). \end{aligned}$$

### Group 8.4.3

$$\langle a, b, c, d \mid [b, a], [d, b][c, a] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $(p - 1)(p^2 - 1)(p^2 - p)p^{20}$ . The nucleus has dimension 12. When  $p = 5$  there are 310 orbits of subspace of the nucleus of codimension 1, and when  $p > 5$  the number of orbits is

$$p^3 + 3p^2 + 12p + (p + 4) \gcd(p - 1, 3) + 2 \gcd(p - 1, 4) + \gcd(p - 1, 5) + 31.$$

The total number of descendants of order  $p^9$  with exponent  $p$  is 7 more than the number of these orbits.

### Group 8.4.4

$$\langle a, b, c, d \mid [d, b][c, a], [d, c][b, a]^\omega \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p$ , and the automorphism group has order  $2(p^4 - 1)(p^4 - p^2)p^{16}$ . The nucleus has dimension 12. The number of orbits of subspaces of the nucleus of codimension 1 is

$$\frac{1}{2}(p^4 + 3p^2 + 5p + 11) + \frac{1}{4}(\gcd(p^2 + 1, 5) - 1).$$

The number of descendants of order  $p^9$  with exponent  $p$  is equal to the number of these orbits.

## 8.5 Five generator groups

### Group 8.5.1

$$\langle a, b, c, d, e \mid [e, a], [c, b], [d, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^6 + p^5 - p^3$ , and the automorphism group has order  $(p-1)^2(p^3-1)(p^3-p)(p^3-p^2)p^{22}$ . The nucleus has dimension 9, and there are 8 orbits of subspaces of the nucleus of codimension 1. There are 28 descendants of order  $p^9$  with exponent  $p$ .

### Group 8.5.2

$$\langle a, b, c, d, e \mid [d, a], [e, a], [d, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^6 + p^5 - p^3$ , and the automorphism group has order  $(p^2-1)(p^2-p)(p^3-1)(p^3-p)(p^3-p^2)p^{21}$ . The nucleus has dimension 8 and there are  $p+4+\gcd(p-1, 3)$  orbits of subspaces of the nucleus of codimension 1. There are  $2p+14+2\gcd(p-1, 3)$  descendants of order  $p^9$  with exponent  $p$ .

### Group 8.5.3

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $3p^5 - 3p^3 + p^2$ , and the automorphism group has order  $2(p-1)^5p^{23}$ . The nucleus has dimension 8, and there are  $p+26$  orbits of subspaces of the nucleus of codimension 1. There are  $2p+101$  descendants of order  $p^9$  with exponent  $p$ .

### Group 8.5.4

$$\langle a, b, c, d, e \mid [d, a] = [c, b], [e, a], [d, b], [e, b], [d, c], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $2p^5 + p^4 - 2p^3$ , and the automorphism group has order  $(p-1)^2(p^2-1)(p^2-p)p^{24}$ . The nucleus has dimension 8, and there are  $p+10$  orbits of subspaces of the nucleus of codimension 1. There are  $2p+37$  descendants of order  $p^9$  with exponent  $p$ .

### Group 8.5.5

$$\langle a, b, c, d, e \mid [c, a], [d, a] = [c, b], [e, a], [d, b], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $2p^5 + p^4 - 2p^3$ , and the automorphism group has order  $(p^2 - 1)^2(p^2 - p)^2p^{19}$ . The nucleus has dimension 8, and there are  $2p + 10 + \gcd(p - 1, 3)$  orbits of subspaces of the nucleus of codimension 1. There are  $5p + 35 + 3 \gcd(p - 1, 3)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.6

$$\langle a, b, c, d, e \mid [d, b] = [c, a]^\omega, [d, c] = [b, a], [e, a], [c, b], [e, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $2(p - 1)(p^2 - 1)^2p^{23}$ . The nucleus has dimension 8, and there are  $p + 5$  orbits of subspaces of the nucleus of codimension 1. There are  $2p + 17$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.7

$$\langle a, b, c, d, e \mid [e, b] = [c, a][d, b]^m, [d, c] = [b, a], [e, c] = [d, b], [d, a], [e, a], [c, b], [e, d] \rangle,$$

where  $m$  is chosen so that  $x^3 + mx - 1$  is irreducible over  $\text{GF}(p)$ . Different choices of  $m$  give isomorphic groups.

The number of conjugacy classes is  $p^5 + p^4 - p$ , and the automorphism group has order  $3(p - 1)(p^3 - 1)p^{18}$ . The nucleus has dimension 6, and there are 2 orbits of subspaces of the nucleus of codimension 1. The number of descendants of order  $p^9$  with exponent  $p$  is

$$\frac{p^2 + p + 10 + 2 \gcd(p - 1, 3)}{3}.$$

#### Group 8.5.8

$$\langle a, b, c, d, e \mid [d, a][c, b]^\omega, [e, a], [e, b] = [c, a], [d, b], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p$ , and the automorphism group has order  $2(p - 1)(p^2 - 1)p^{18}$ . The nucleus has dimension 5, and there are  $p + 4$  orbits of subspaces of the nucleus of codimension 1. There are  $\frac{5}{2}(p + 1) + 9$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.9

$$\langle a, b, c, d, e \mid [d, a], [e, a], [e, b] = [c, a], [d, b], [d, c] = [b, a], [e, c], [e, d] = [c, b] \rangle$$

The number of conjugacy classes is  $p^5 + p^4 - p$ , and the automorphism group has order  $(p+1)(p-1)^2 p^{16}$ . The nucleus has dimension 5, and there are  $2p+6 + \gcd(p-1, 3)$  orbits of subspaces of the nucleus of codimension 1. There are  $4p+17 + 2 \gcd(p-1, 3)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.10

$$\langle a, b, c, d, e \mid [d, a], [e, a], [d, b], [e, b] = [c, a], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $(p-1)(p^2-1)(p^2-p)p^{21}$ . The nucleus has dimension 6, and there are  $p+5$  orbits of subspaces of the nucleus of codimension 1. There are  $2p+20$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.11

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [e, b] = [c, a], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $(p-1)^3 p^{21}$ . The nucleus has dimension 6, and there are 9 orbits of subspaces of the nucleus of codimension 1. There are 37 descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.12

$$\langle a, b, c, d, e \mid [e, a], [c, b], [d, b], [e, c], [e, d], [d, c] = [b, a], [e, b] = [c, a] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $(p-1)^3 p^{19}$ . The nucleus has dimension 5, and there are 9 orbits of subspaces of the nucleus of codimension 1. There are  $p+33 + \gcd(p-1, 3)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.13

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [e, b][d, b]^\omega = [c, a], [d, c] = [b, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $2(p-1)^2(p^2-1)p^{18}$ . The nucleus has dimension 6, and there are 11 orbits of subspaces of the nucleus of codimension 1. The number of descendants of order  $p^9$  with exponent  $p$  is

$$\frac{p^2 + 4p + 69}{2}.$$

#### Group 8.5.14

$$\langle a, b, c, d, e \mid [c, b], [d, b], [e, c], [e, d], [d, c] = [b, a], [e, b] = [c, a], [e, a] = [d, a]^\omega \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $2(p-1)(p^2-1)p^{17}$ . The nucleus has dimension 5, and there are 6 orbits of subspaces of the nucleus of codimension 1. There are  $\frac{1}{2}(p+37)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.15

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [d, b], [e, c], [d, c] = [b, a], [e, b] = [c, a] \rangle$$

The number of conjugacy classes is  $p^5 + 2p^4 - p^3 - p^2$ , and the automorphism group has order  $2(p-1)^3p^{17}$ . The nucleus has dimension 5, and there are 12 orbits of subspaces of the nucleus of codimension 1. There are  $\frac{1}{2}(p+81)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.16

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [d, b], [e, b] = [c, a], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 3p^4 - 2p^3 - 2p^2 + p$ , and the automorphism group has order  $(p-1)^4p^{19}$ . The nucleus has dimension 6, and there are 18 orbits of subspaces of the nucleus of codimension 1. There are  $p + 83 + \gcd(p-1, 3)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.17

$$\langle a, b, c, d, e \mid [e, b] = [c, a], [d, c] = [b, a][d, a]^{-1}, [e, a], [c, b], [d, b], [e, c], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 3p^4 - 2p^3 - 2p^2 + p$ , and the automorphism group has order  $2(p-1)^3p^{18}$ . The nucleus has dimension 5, and there are  $p+8$  orbits of subspaces of the nucleus of codimension 1. There are  $\frac{1}{2}(5p+63)$  descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.18

$$\langle a, b, c, d, e \mid [e, c][e, b], [c, a], [d, a], [e, a], [c, b], [d, b], [e, d] \rangle$$

The number of conjugacy classes is  $p^5 + 4p^4 - 3p^3 - 3p^2 + 2p$ , and the automorphism group has order  $6(p-1)^4p^{18}$ . The nucleus has dimension 6, and there are 12 orbits of subspaces of the nucleus of codimension 1. The number of descendants of order  $p^9$  with exponent  $p$  is

$$\frac{1}{6}(p^2 + 10p + 241 + 2 \gcd(p-1, 3)).$$

#### Group 8.5.19

$$\langle a, b, c, d, e \mid [d, a], [e, a], [c, b], [d, b], [e, b], [d, c], [e, c] \rangle$$

The number of conjugacy classes is  $2p^5 + 2p^4 - 3p^3 - p^2 + p$ , and the automorphism group has order  $(p-1)(p^2-1)^2(p^2-p)^2p^{17}$ . The nucleus has dimension 7, and there are 11 orbits of subspaces of the nucleus of codimension 1. There are 40 descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.20

$$\langle a, b, c, d, e \mid [d, c] = [b, a], [c, a], [d, a], [e, a], [d, b], [e, b], [e, d] \rangle$$

The number of conjugacy classes is  $2p^5 + p^4 - 2p^3$ , and the automorphism group has order  $(p-1)^4p^{23}$ . The nucleus has dimension 7, and there are 15 orbits of subspaces of the nucleus of codimension 1. There are 67 descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.5.21

$$\langle a, b, c, d, e \mid [d, c] = [b, a], [c, a], [d, a], [e, a], [c, b], [d, b], [e, d] \rangle$$

The number of conjugacy classes is  $2p^5 + p^4 - 2p^3$ , and the automorphism group has order  $(p-1)^2(p^2-1)(p^2-p)p^{20}$ . The nucleus has dimension 6,

and there are 10 orbits of subspaces of the nucleus of codimension 1. There are 45 descendants of order  $p^9$  with exponent  $p$ .

### Group 8.5.22

$$\langle a, b, c, d, e \mid [d, c] = [b, a], [c, a], [d, a], [e, a], [c, b], [d, b], [e, b] \rangle$$

The number of conjugacy classes is  $2p^5 + p^4 - 2p^3$ , and the automorphism group has order  $(p^2 - 1)^2(p^2 - p)^2p^{17}$ . The nucleus has dimension 5, and there are 5 orbits of subspaces of the nucleus of codimension 1. There are  $p + 17$  descendants of order  $p^9$  with exponent  $p$ .

## 8.6 Six generator groups

For all these groups we take the generators to be  $a, b, c, d, e, f$ , and we just give the relations, with the class two and exponent  $p$  conditions understood.

### Group 8.6.1

$$\begin{aligned} [c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] \end{aligned}$$

The number of conjugacy classes is  $2p^6 - p^4$  and the order of the automorphism group is  $(p - 1)(p^2 - 1)(p^2 - p)(p^3 - 1)(p^3 - p)(p^3 - p^2)p^{23}$ . The nucleus has dimension 5, and there are 5 orbits of subspaces of the nucleus of codimension 1. There are 28 descendants of order  $p^9$  with exponent  $p$ .

### Group 8.6.2

$$\begin{aligned} [c, b], [d, a], [d, b] = [b, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] \end{aligned}$$

The number of conjugacy classes is  $p^6 + 2p^5 - p^4 - 2p^3 + p^2$  and the order of the automorphism group is  $2(p^2 - 1)^3(p^2 - p)^3p^{20}$ . The nucleus has dimension 4, and there are 2 orbits of subspaces of the nucleus of codimension 1. There are 19 descendants of order  $p^9$  with exponent  $p$ .

### Group 8.6.3

$$[c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is  $p^6 + p^5 - p^3$  and the order of the automorphism group is  $(p-1)(p^2-1)^2(p^2-p)^2p^{24}$ . The nucleus has dimension 4, and there are 2 orbits of subspaces of the nucleus of codimension 1. There are 21 descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.6.4

$$[c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega, [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is  $p^6 + p^4 - p^2$  and the order of the automorphism group is  $2(p^4 - 1)(p^4 - p^2)(p^2 - 1)(p^2 - p)p^{20}$ . The nucleus has dimension 4, and there is only 1 orbit of subspaces of the nucleus of codimension 1. There are 7 descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.6.5

$$[c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is  $p^6 + p^5 - p^3$  and the order of the automorphism group is  $(p-1)^3(p^2-1)(p^2-p)p^{22}$ . The nucleus has dimension 2, and there are 2 orbits of subspaces of the nucleus of codimension 1. There are 31 descendants of order  $p^9$  with exponent  $p$ .

#### Group 8.6.6

$$[c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b], [f, c], [f, d], [f, e]$$

The number of conjugacy classes is  $p^6 + p^4 - p^2$  and the order of the automorphism group is  $(p-1)^2(p^2-1)(p^2-p)p^{21}$ . The nucleus has dimension 1, and there are 10 descendants of order  $p^9$  with exponent  $p$ .



**Group 8.6.7**

$$\begin{aligned}
&[c, a], [c, b], [d, a], [d, b], [e, a], [e, b], [e, c], \\
&[e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [b, a]
\end{aligned}$$

The number of conjugacy classes is  $p^6 + p^5 - p^4 + p^3 - 2p + 1$  and the order of the automorphism group is  $(p^4 - 1)(p^4 - p^3)(p^2 - 1)^2(p^2 - p)p^{13}$ . The nucleus has dimension 2, and there is only one orbit of subspaces of the nucleus of codimension 1. There are 10 descendants of order  $p^9$  with exponent  $p$ .

**Group 8.6.8**

$$\begin{aligned}
&[c, a], [c, b], [d, a], [d, b], [e, a], [e, b], [e, c], \\
&[e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [b, a][d, c]
\end{aligned}$$

The number of conjugacy classes is  $p^6 + 3p^3 - 2p^2 - 3p + 2$  and the order of the automorphism group is  $6(p^2 - 1)^3(p^2 - p)p^{14}$ . This group is terminal.

**Group 8.6.9**

$$\begin{aligned}
&[c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], \\
&[e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [b, a]
\end{aligned}$$

The number of conjugacy classes is  $p^6 + 2p^3 - p^2 - 2p + 1$  and the order of the automorphism group is  $(p^2 - 1)^2(p^2 - p)^2p^{15}$ . This group is terminal.

**Group 8.6.10**

$$\begin{aligned}
&[c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], \\
&[e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [c, a]
\end{aligned}$$

The number of conjugacy classes is  $p^6 + p^5 - p^4 + p^2 - p$  and the order of the automorphism group is  $(p^2 - 1)^2(p^2 - p)^2p^{20}$ . The nucleus has dimension 2, and there is only 1 orbit of subspaces of the nucleus of codimension 1. There are 13 descendants of order  $p^9$  with exponent  $p$ .

**Group 8.6.11**

$$[c, b], [d, a], [d, b] = [c, a], [d, c] = [b, a]^\omega, [e, a], [e, b], [e, c], \\ [e, d], [f, a], [f, b], [f, c], [f, d], [f, e] = [b, a]$$

The number of conjugacy classes is  $p^6 + p^3 - p$  and the order of the automorphism group is  $2(p^4 - 1)(p^3 - p^2)(p^2 - 1)p^{13}$ . This group is terminal.

### Group 8.6.12

$$[c, b], [d, a], [d, b], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b], [f, c], [f, d] = [c, a], [f, e]$$

The number of conjugacy classes is  $p^6 + p^4 - p^2$  and the order of the automorphism group is  $(p^2 - 1)^2(p^2 - p)^2p^{18}$ . This group is terminal.

### Group 8.6.13

$$[b, a], [d, a], [e, a][c, a], [f, a], [c, b], [d, b] = [c, a], [e, b], \\ [f, b][c, a]^2, [d, c], [e, c], [e, d] = [f, c], [f, d], [f, e] = [c, a][f, c]$$

The number of conjugacy classes is  $p^6 + p^3 - p$  and the order of the automorphism group is  $(p - 1)(p^2 - 1)(p^2 - p)p^{19}$ . This group is terminal.

### Group 8.6.14

$$[c, b], [d, a], [d, b] = [c, a], [d, c], [e, a], [e, b], [e, c], \\ [e, d] = [b, a], [f, a], [f, b] = [c, a]^m, [f, c] = [b, a], [f, d], [f, e] = [c, a],$$

where  $x^3 - mx + 1$  is irreducible over  $\text{GF}(p)$ . (Different choices of  $m$  give isomorphic groups.)

The number of conjugacy classes is  $p^6 + p^2 - 1$  and the order of the automorphism group is  $3(p^6 - 1)(p - 1)p^{15}$ . This group is terminal.

## 8.7 Seven generator groups

### Group 8.7.1

$$\langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle \times \langle f \rangle \times \langle g \rangle.$$

The number of conjugacy classes is  $p^7 + p^6 - p^5$ , and the automorphism group has order  $(p^8 - p^6)(p^8 - p^7)(p^6 - p)(p^6 - p^2)(p^6 - p^3)(p^6 - p^4)(p^6 - p^5)$ . The nucleus has dimension 2, and there is only one orbit of subspaces of the nucleus of dimension 1. There are 6 descendants of order  $p^9$  with exponent  $p$ .

### Group 8.7.2

$$\langle a, b \rangle \times_{[b,a]=[d,c]} \langle c, d \rangle \times \langle e \rangle \times \langle f \rangle \times \langle g \rangle.$$

The number of conjugacy classes is  $p^7 + p^4 - p^3$ , and the automorphism group has order  $(p^8 - p^4)(p^8 - p^7)(p^6 - p^4)p^5(p^4 - p)(p^4 - p^2)(p^4 - p^3)$ . This group is terminal.

### Group 8.7.3

$$\langle a, b \rangle \times_{[b,a]=[d,c]=[f,e]} \langle c, d \rangle \times_{[b,a]=[d,c]=[f,e]} \langle e, f \rangle \times \langle g \rangle.$$

The number of conjugacy classes is  $p^7 + p^2 - p$ , and the automorphism group has order  $(p^8 - p^2)(p^8 - p^7)(p^6 - p^2)p^5(p^4 - p^2)p^3(p^2 - p)$ . This group is terminal.

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